LEARNING OBJECTIVES

In this section, you will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

1.3 RADICALS AND RATIONAL EXPONENTS

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in **Figure 1**, and use the Pythagorean Theorem.

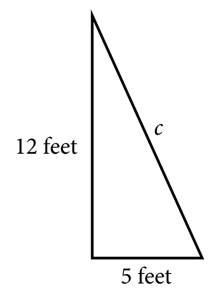


Figure 1

$$a^2 + b^2 = c^2$$
$$5^2 + 12^2 = c^2$$
$$169 = c^2$$

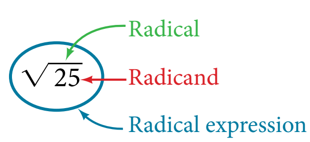
Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.



principal square root

Th **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a **radical expression**, with a symbol called a **radical** over the term called the **radicand**: $\sqrt{a}$.

Q & A...

Does $\sqrt{25} = \pm 5$?

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

Example 1 Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{100}$ **b.** $\sqrt{\sqrt{16}}$ **c.** $\sqrt{25 + 144}$ **d.** $\sqrt{49} - \sqrt{81}$

Solution

a. $\sqrt{100} = 10$ because $10^2 = 100$

b. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$

c. $\sqrt{25 + 144} = \sqrt{169} = 13$ because $13^2 = 169$

d. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Q & A...

For $\sqrt{25 + 144}$, can we find the square roots before adding?

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

Try It #1

a. $\sqrt{225}$ **b.** $\sqrt{\sqrt{81}}$ **c.** $\sqrt{25 - 9}$ **d.** $\sqrt{36} + \sqrt{121}$

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots*, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

the product rule for simplifying square roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b.

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

How To...

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example 2 Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$ **b.** $\sqrt{162a^5b^4}$

Solution

a. $\sqrt{100 \cdot 3}$	Factor perfect square from radicand.
$\sqrt{100} \cdot \sqrt{3}$	Write radical expression as product of radical expressions.
$10\sqrt{3}$	Simplify.

b. $\sqrt{81a^4b^4 \cdot 2a}$ — Factor perfect square from radicand.

$\sqrt{81a^4b^4} \cdot \sqrt{2a}$ — Write radical expression as product of radical expressions.

$9a^2b^2\sqrt{2a}$ — Simplify.

Try It #2

Simplify $\sqrt{50x^2y^3z}$.

How To…

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example 3 **Using the Product Rule to Simplify the Product of Multiple Square Roots**

Simplify the radical expression. $\sqrt{12} \cdot \sqrt{3}$

Solution

$\sqrt{12 \cdot 3}$ — Express the product as a single radical expression.

$\sqrt{36}$ — Simplify.

6

Try It #3

Simplify $\sqrt{50x} \cdot \sqrt{2x}$ assuming $x > 0$.

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots*. It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately.

We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

the quotient rule for simplifying square roots

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

How To…

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expression.
2. Simplify the numerator and denominator.

Example 4 **Using the Quotient Rule to Simplify Square Roots**

Simplify the radical expression. $\sqrt{\frac{5}{36}}$

Solution

$\frac{\sqrt{5}}{\sqrt{36}}$ — Write as quotient of two radical expressions.

$\frac{\sqrt{5}}{6}$ — Simplify denominator.

Try It #4

Simplify $\sqrt{\dfrac{2x^2}{9y^4}}$.

Example 5 Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression. $\dfrac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

Solution

$\sqrt{\dfrac{234x^{11}y}{26x^7y}}$	Combine numerator and denominator into one radical expression.
$\sqrt{9x^4}$	Simplify fraction.
$3x^2$	Simplify square root.

Try It #5

Simplify $\dfrac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2} + \sqrt{18} = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$.

How To...

Given a radical expression requiring addition or subtraction of square roots, solve.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example 6 Adding Square Roots

Add $5\sqrt{12} + 2\sqrt{3}$.

Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4 \cdot 3}$. According the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5(2)\sqrt{3}$, which is $10\sqrt{3}$. Now the terms have the same radicand so we can add.

$10\sqrt{3} + 2\sqrt{3} = 12\sqrt{3}$

Try It #6

Add $\sqrt{5} + 6\sqrt{20}$.

Example 7 Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c} - 14\sqrt{8a^3b^4c}$.

Solution

Rewrite each term so they have equal radicands.

$$\begin{aligned}20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\ &= 20(3)(2)|a|b^2\sqrt{2ac}\\ &= 120|a|b^2\sqrt{2ac}\\ 14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c}\\ &= 14(2)|a|b^2\sqrt{2ac}\\ &= 28|a|b^2\sqrt{2ac}\end{aligned}$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac} - 28|a|b^2\sqrt{2ac} = 92|a|b^2\sqrt{2ac}$$

Try It #7

Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator.*

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a + b\sqrt{c}$, then the conjugate is $a - b\sqrt{c}$.

How To...

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Example 8 Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$\frac{2\sqrt{30}}{30}$$

$$\frac{\sqrt{30}}{15}$$

Try It #8

Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

How To...

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 9 Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$\frac{4-4\sqrt{5}}{-4} \qquad \text{Use the distributive property.}$$

$$\sqrt{5}-1 \qquad \text{Simplify.}$$

Try It #9

Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Using Rational Roots

Although square roots are the most common rational roots, we can also find cuberoots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3 = 8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3 = 8$, we say that 2 is the cube root of 8.

Th nth root of a is a number that, when raised to the nth power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5 = -243$. If a is a real number with at least one nth root, then the **principal *n*th root** of a is the number with the same sign as a that, when raised to the nth power, equals a.

The principal nth root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the **index** of the radical.

principal nth root

If a is a real number with at least one nth root, then the **principal *n*th root** of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the nth power, equals a. The **index** of the radical is n.

Example 10 Simplifying *n*th Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$ **b.** $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$ **c.** $-\sqrt[3]{\dfrac{8x^6}{125}}$ **d.** $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

b. First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$

c. $\dfrac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}}$ Write as quotient of two radical expressions.

$\dfrac{-2x^2}{5}$ Simplify.

d. $8\sqrt[4]{3} - 2\sqrt[4]{3}$ Simplify to get equal radicands.

$6\sqrt[4]{3}$ Add.

Try It #10

Simplify.

a. $\sqrt[3]{-216}$ **b.** $\dfrac{3\sqrt[4]{80}}{\sqrt[4]{5}}$ **c.** $6\sqrt[3]{9,000} + 7\sqrt[3]{576}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

rational exponents

Rational exponents are another way to express principal nth roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

How To...

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 11 Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution

The 2 tells us the power and the 3 tells us the root.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

Try It #11

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Example 12 Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$.

Try It #12

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Example 13 Simplifying Rational Exponents

Simplify:

a. $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$ **b.** $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Solution

a. $30x^{\frac{3}{4}}x^{\frac{1}{5}}$ Multiply the coeffici t.

$30x^{\frac{3}{4}+\frac{1}{5}}$ Use properties of exponents.

$30x^{\frac{19}{20}}$ Simplify.

b. $\left(\frac{9}{16}\right)^{\frac{1}{2}}$ Use defin tion of negative exponents.

$\sqrt{\frac{9}{16}}$ Rewrite as a radical.

$\frac{\sqrt{9}}{\sqrt{16}}$ Use the quotient rule.

$\frac{3}{4}$ Simplify.

Try It #13

Simplify $8x^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$.

Access these online resources for additional instruction and practice with radicals and rational exponents.

- Radicals (http://openstaxcollege.org/l/introradical)
- Rational Exponents (http://openstaxcollege.org/l/rationexpon)
- Simplify Radicals (http://openstaxcollege.org/l/simpradical)
- Rationalize Denominator (http://openstaxcollege.org/l/rationdenom)

1.3 SECTION EXERCISES

VERBAL

1. What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.

2. Where would radicals come in the order of operations? Explain why.

3. Every number will have two square roots. What is the principal square root?

4. Can a radical with a negative radicand have a real square root? Why or why not?

NUMERIC

For the following exercises, simplify each expression.

5. $\sqrt{256}$

6. $\sqrt{\sqrt{256}}$

7. $\sqrt{4(9+16)}$

8. $\sqrt{289} - \sqrt{121}$

9. $\sqrt{196}$

10. $\sqrt{1}$

11. $\sqrt{98}$

12. $\sqrt{\frac{27}{64}}$

13. $\sqrt{\frac{81}{5}}$

14. $\sqrt{800}$

15. $\sqrt{169} + \sqrt{144}$

16. $\sqrt{\frac{8}{50}}$

17. $\frac{18}{\sqrt{162}}$

18. $\sqrt{192}$

19. $14\sqrt{6} - 6\sqrt{24}$

20. $15\sqrt{5} + 7\sqrt{45}$

21. $\sqrt{150}$

22. $\sqrt{\frac{96}{100}}$

23. $(\sqrt{42})(\sqrt{30})$

24. $12\sqrt{3} - 4\sqrt{75}$

25. $\sqrt{\frac{4}{225}}$

26. $\sqrt{\frac{405}{324}}$

27. $\sqrt{\frac{360}{361}}$

28. $\frac{5}{1+\sqrt{3}}$

29. $\frac{8}{1-\sqrt{17}}$

30. $\sqrt[4]{16}$

31. $\sqrt[3]{128} + 3\sqrt[3]{2}$

32. $\sqrt[5]{\frac{-32}{243}}$

33. $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

34. $3\sqrt[3]{-432} + \sqrt[3]{16}$

ALGEBRAIC

For the following exercises, simplify each expression.

35. $\sqrt{400x^4}$

36. $\sqrt{4y^2}$

37. $\sqrt{49p}$

38. $(144p^2q^6)^{\frac{1}{2}}$

39. $m^{\frac{5}{2}}\sqrt{289}$

40. $9\sqrt{3m^2} + \sqrt{27}$

41. $3\sqrt{ab^2} - b\sqrt{a}$

42. $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

43. $\sqrt{\frac{225x^3}{49x}}$

44. $3\sqrt{44z} + \sqrt{99z}$

45. $\sqrt{50y^8}$

46. $\sqrt{490bc^2}$

47. $\sqrt{\frac{32}{14d}}$

48. $q^{\frac{3}{2}}\sqrt{63p}$

49. $\frac{\sqrt{8}}{1-\sqrt{3x}}$

50. $\sqrt{\frac{20}{121d^4}}$

51. $w^{\frac{3}{2}}\sqrt{32} - w^{\frac{3}{2}}\sqrt{50}$

52. $\sqrt{108x^4} + \sqrt{27x^4}$

53. $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

54. $\sqrt{147k^3}$

55. $\sqrt{125n^{10}}$

56. $\sqrt{\frac{42q}{36q^3}}$

57. $\sqrt{\frac{81m}{361m^2}}$

58. $\sqrt{72c} - 2\sqrt{2c}$

59. $\sqrt{\dfrac{144}{324d^2}}$

60. $\sqrt[3]{24x^6} + \sqrt[3]{81x^6}$

61. $\sqrt[4]{\dfrac{162x^6}{16x^4}}$

62. $\sqrt[3]{64y}$

63. $\sqrt[3]{128z^3} - \sqrt[3]{-16z^3}$

64. $\sqrt[5]{1{,}024c^{10}}$

REAL-WORLD APPLICATIONS

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to fi d the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90{,}000 + 160{,}000}$. What is the length of the guy wire?

66. A car accelerates at a rate of $6 - \dfrac{\sqrt{4}}{\sqrt{t}}\ \text{m}/s^2$ where t is the time in seconds after the car moves from rest. Simplify the expression.

EXTENSIONS

For the following exercises, simplify each expression.

67. $\dfrac{\sqrt{8} - \sqrt{16}}{4 - \sqrt{2}} - 2^{\frac{1}{2}}$

68. $\dfrac{4^{\frac{3}{2}} - 16^{\frac{3}{2}}}{8^{\frac{1}{3}}}$

69. $\dfrac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}} \cdot \dfrac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

70. $\dfrac{a}{a - \sqrt{c}}$

71. $\dfrac{x\sqrt{64y} + 4\sqrt{y}}{\sqrt{128y}}$

72. $\left(\dfrac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\dfrac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\dfrac{\sqrt[3]{64} + \sqrt[4]{256}}{\sqrt{64} + \sqrt{256}}}$